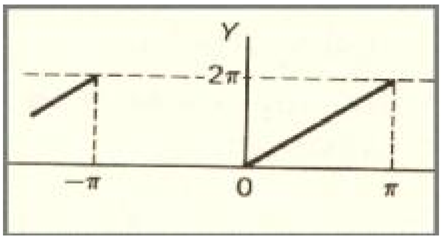
Lecture # 23

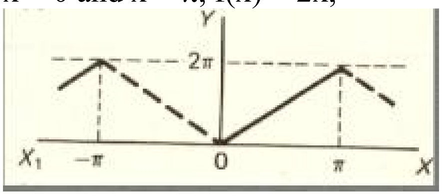
Half-Range Series



Sometime a function of period 2π is defined over the range 0 to π, instead of the normal $-\pi$ to π, or 0 to 2π. We then have a choice of how to proceed.

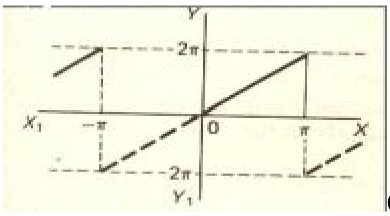
For example, if we are told between

$x = 0$ and $x = \pi$, $f(x) = 2x$,



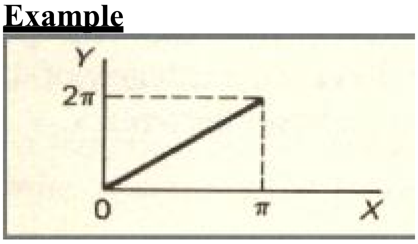
then, since the period is 2π, we have no evidence of how the function behave between $x = -\pi$ and $x = 0$.

If the waveform were as shown in (a), the function would be an even function, symmetrical about the y-axis and the series would have only cosine terms (including possibly a_0).



On the other hand, if the waveform were as shown in (b), the function would be odd, being symmetrical about the origin and the series would have only sine terms.

Example



A function f(x) is defined by

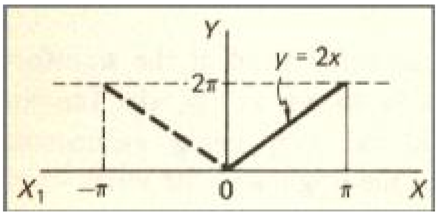
$f(x) = 2x \quad 0 < x < \pi$

$f(x) = f(x + 2\pi)$

Obtain a half-range cosine series to represents the function.

To obtain a cosine series, i.e. a series with no sine terms, we need an even function.

Therefore, we assume the waveform between $x = -\pi$ and $x = 0$ to be as shown, making the total graph symmetrical about the y-axis.



Now we can find expressions for the Fourier coefficients as usual.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} 2x\,dx = \frac{2}{\pi}\left[x^2\right]_0^{\pi} = 2\pi$$

$\therefore\ a_0 = 2\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} 2x\cos nx\,dx = \frac{4}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{4}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right\}$$

$$= \frac{4}{\pi}\left\{(0-0) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\right\} = \left(\frac{4}{\pi n^2}\right)(\cos n\pi - 1)$$

$\cos n\pi = 1$ (n even) and $\cos n\pi = -1$ (n odd)

$\therefore\ a_n = 0$ (n even) and $a_n = -\frac{8}{\pi n^2}$ (n odd)

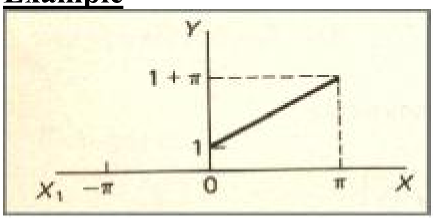
All that now remains is b_n which is zero, since f(x) is an even function, i.e.

$b_n = 0$ So $a_0 = 2\pi$, $a_n = 0$ (n even) and $a_n = -\frac{8}{\pi n^2}$ (n odd),

$b_n = 0$. Therefore

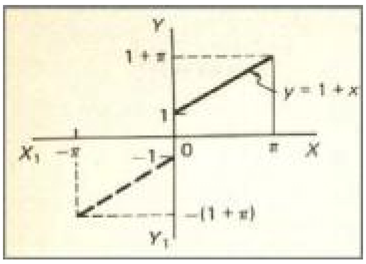
$$f(x) = \pi - \frac{8}{\pi}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \ldots\right\}$$

Example



Determine a half-range sine series to represent the function f(x) defined by $f(x) = 1 + x \quad 0 < x < \pi$

$f(x) = f(x + 2\pi)$



We choose the waveform between $x = -\pi$ and $x = 0$ so that the graph is symmetrical about the origin. The function is then an odd function and the series will contain only sine terms.

$\therefore\ a_0 = 0$ and $a_n = 0$

π

$\pi b_n = \frac{1}{}$

$\pi b_n = \frac{2}{}$

2

$\int$ f(x) sin nx dx

$-\pi$

π

$\int_0$ f(x) sin nx dx

π

$\underline{2}$ ⌈‖ ⌈

$\underline{-\cdots\text{cosnx}}$] π $\underline{1}$ ⌉|

π⌊ $b_n = \int$

0

(1+x)sin nx dx = π { | | ⌊ (1 + x)

n |$_0$ + $\int$

cosnxdx}

| ⌋

$\underline{2}$ ⌈ $\underline{1+\pi}$ π

$\underline{1}$ $\underline{1}$ ⌈ $\underline{\text{sinnx}}$| ⌉

$\underline{2}$ ⌈| $\underline{1}$

$\underline{1+\pi}$

|⌉ $\underline{2}$

$\pi =$ { $-$

⌊

n cos nπ + n + n | ⌊ n

| } =

0⌋

π| ⌊ n^{-}

n cos nπ} | = πn

{1 − (1 + π) cos nπ}

⌋|⌋ cos nπ = 1 (n even) and cos nπ = $-$ 1 (n odd)

n $\therefore\ b_n = -\frac{2}{}$

π

$\therefore \quad = \underline{4+2\pi}\ n$

(n even)

(n odd)

Substituting in the general expression $f(x) = \sum_{x=1}^{\infty} b_n \sin nx$ we have

$$f(x)=\frac{4+2\pi}{\pi}\left\{\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots\right\}-2\left\{\frac{1}{2}\sin 2x+\frac{1}{4}\sin 4x+\frac{1}{6}\sin 6x+\ \ \right)$$

and the required series obtained

$(\ \underline{4})$ $\underline{1}$ $\underline{1}$ $\underline{1}$ $\underline{1}$ $\underline{1}$

$$f(x) = \left(\frac{\;}{\pi} + 2 \right) \{\sin x + \frac{\;}{3} \sin 3x + \frac{\;}{5} \sin 5x + ..\} - 2\{\frac{\;}{2} \sin 2x + \frac{\;}{4} \sin 4x + \frac{\;}{6} \sin 6x + \qquad \}$$

So knowledge of odd and even functions and of half-range series saves a deal of unnecessary work on occasions.